The problem of camouflaging via mirror reflections

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Abstract

This work is related to billiards and their applications in geometric optics. It is known that perfectly invisible bodies with mirror surface do not exist. It is therefore natural to search for bodies that are, in a sense, close to invisible. We introduce a *visibility index* of a body measuring the mean angle of deviation of incident light rays, and derive a lower estimate for this index. This estimate is a function of the body's volume and of the minimal radius of a ball containing the body. This result is far from being final and opens a possibility for further research.

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1 Introduction

The idea of invisibility has always been attractive for the people. Stories on magic cap and cloak of invisibility form an essential part of folklore, myths and fairy tales. Methods of camouflaging establishments, troops and other objects of importance are of great interest for military in all times; one of the most famous developments of the 20th century in this area is the Stealth technology aiming at making airplanes invisible for radars of the enemy.

In the last decades intensive work has been carried on developing technology of meta-materials possessing unusual properties (see, e.g., [12]) having in mind, in particular, creating something like a transparent meta-material cover with varying refractive index that makes invisible every object placed inside.

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An important and interesting mathematical construction in the 2D case is proposed in the paper by Leonhardt [10]. It describes how to make an object invisible by wrapping a lens (a transparent material with varying refractive index) around it.

There is an interesting question, to what extent can one create the effect of invisibility, if only mirror systems are allowed to use. Mirrors are much easier and cheaper for fabrication than hypothetical meta-material structures, and even than traditional lenses with controlled refractive index. Some results in this direction have already been obtained. There exist and are described (connected) bodies invisible from 1 point [17, 14] and (infinitely connected) bodies invisible from 2 points [19]. There exist (connected and even simply connected) bodies invisible in 1 direction (that is, from an infinitely distant point) [1], (finitely connected) bodies invisible in 2 directions [16], as well as (infinitely connected) bodies invisible in 3 [18] and (in the 2D case) in n directions, where the number $n \in \mathbb{N}$ of directions is arbitrary [15].

On the other hand, there are negative results revealing restricted possibilities of mirror systems as compared with more sophisticated technologies. In particular, non-existence of perfectly invisible bodies (that is those that are invisible in any direction or (equivalently) from any point outside the body) is proved in [16]. Further, a conjecture proposed in [14] states that the set of light rays that are invisible for any fixed body has measure zero. This conjecture is closely connected with the long-standing Ivrii's conjecture [8] stating that the measure of the set of periodic billiard trajectories in a bounded domain has measure zero. If Ivrii's conjecture is true then, most probably, true also is the conjecture on invisible light rays.

At the moment Ivrii's conjecture is proved only for trajectories with 3 [20, 21] and 4 [7, 6] reflections. The corresponding invisibility conjecture for the case of 3 reflections is easily obtained by slightly rephrasing the proof of Ivrii's conjecture with 3 reflections. A unified approach developed by Glutsyuk in [6] based on complexification of billiards allows one to derive both Ivrii's and invisibility conjectures in the case of 4 reflections from his theorem of classification of 4-reflective complex planar analytic billiards (Theorem 1.7 in [6]). Note that the proof of the conjectures in the case of 4 reflections is much more difficult as compared to the case of 3 reflections.

In real life quite common is the situation when perfect invisibility is impossible to achieve. In such cases one tries to reach the effect of partial invisibility, or camouflaging, when the object, though not disappearing completely, still becomes difficult to detect by an observer. It is natural to set such a question in the framework of mirror invisibility. In order to state a mathematical problem, one needs first to determine an *index of visibility*, a certain positive quantity which is close to zero if the body is, in a sense, difficult to detect. This quantity should never vanish, since perfectly invisible bodies do not exist.

Then one should consider the question, how small can this index be made in a certain class of bodies. For example, if even the index does not vanish, is it possible to construct a sequence of bodies of constant volume with the index going to zero?

Choosing the visibility index is not an easy task; it is more difficult than just defining the notion of invisibility. The body is observed against a certain background, and the choice will depend, in particular, on the distance of the body from the background. In the limit, when the background is infinitely distant, the visibility index is determined by the angles of deviation of light rays from their original directions and does not depend on transverse displacement of the rays. This limit will be used later on in this paper.

The aim of the paper is to give (partial) answers to the questions stated above. If the body has the volume A and is contained in a sphere of radius r, then its visibility index is not less than a certain positive value, a function of A and r. This function goes to zero when A is constant as $r \to \infty$.

2 Main definitions and statement of the results

First of all fix the notation. A body with specular surface is a bounded finitely connected domain with piecewise smooth boundary in Euclidean space $\mathbb{R}^d$ with $d \geq 2$. It will be called a *domain* and designated by D. Since everything is about specular reflections in the framework of geometric optics, we adopt the notation of billiard theory and consider the billiard in $\mathbb{R}^d \setminus D$.

Fix a domain D and take a sphere S_R^{d-1} of radius $R > 0$ centered at the origin and containing D. It is assumed that the background lies on the sphere. As a result of observation of the background one must conclude whether the body is or is not present here. For a point ξ on the sphere and a unit vector v such that $\langle v, \xi \rangle < 0$ consider the trajectory of a billiard particle starting at ξ with the velocity v and the half-line with the endpoint ξ and the directing vector v, and denote by $\theta = \theta_{R,D}(v, \xi)$ the angular distance between the (second) points of intersection of the trajectory and of the half-line with the sphere (see Fig. 1).

We define the measure spaces

$$(S^{d-1} \times S_R^{d-1})_\pm = \{(v, \xi) \in S^{d-1} \times S_R^{d-1} : \pm\langle v, \xi \rangle \geq 0\}$$

equipped with the measure $\mu_R = \mu$ defined by $d\mu(v, \xi) = |\langle v,\, n(\xi) \rangle|\, dv\, d\xi$, where $n(\xi) = \xi/R$ is the outer unit normal to the sphere S_R^{d-1} at the point ξ. Note that the spaces $(S^{d-1} \times S_R^{d-1})_-$ and $(S^{d-1} \times S_R^{d-1})_+$ correspond to billiard trajectories entering the sphere S_R^{d-1} and leaving it, respectively, and μ is a natural measure counting the amount of billiard trajectories intersecting the sphere.

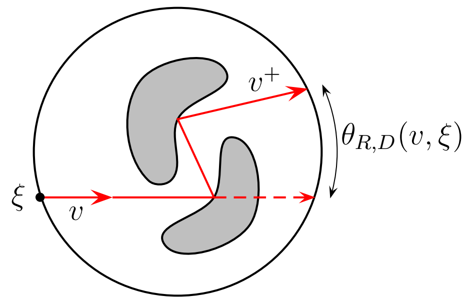

Figure 1: The domain D is shown shaded (it has 2 connected components); $\theta_R(v,\xi)$ indicates the angular deviation of the particle reflected from D with respect to freely moving particle.

Take a monotone increasing function $f : [0,\,\pi] \to \mathbb{R}$ such that $f(0) = 0$ and consider the value

$$\mathcal{F}_R(D) = \int_{(S^{d-1}\times S_R^{d-1})_-} f(\theta_{R,D}(v,\xi))\, d\mu(v,\xi),$$

which will be called the *visibility index* of D.

Note that the equality $\mathcal{F}_R(D) = 0$ does not yet guarantee invisibility of D. In fact the domain D is invisible, if and only if $\mathcal{F}_R(D) = 0$ for any R sufficiently large.

In the limit $R \to \infty$ the quantity θ does not depend on the transverse displacement (shift) of the trajectory going away, but only on the angle between the initial and final velocities. Let us introduce some more notation. For the billiard trajectory entering the sphere S_R^{d-1} at a point ξ and having a velocity v at this point, we denote by

$$v^+ = v_D^+(v,\xi), \quad \xi^+ = \xi_{D,R}^+(v,\xi)$$

the second point of intersection of this trajectory with the sphere (when leaving the sphere S_R^{d-1}) and its velocity at this point. (Note that the velocity v^+ does not depend on the radius R of the sphere containing D.) The mapping

$$T = T_{D,R} : (v,\xi) \mapsto \big(v_D^+(v,\xi), \xi_{D,R}^+(v,\xi)\big)$$

is a one-to-one mapping from $(S^{d-1}\times S_R^{d-1})_-$ onto $(S^{d-1}\times S_R^{d-1})_+$ (defined up to a subset of measure zero) preserving the measure μ.

Then in the limit mentioned above, θ is the angle between v and v^+, $\theta = \arccos\langle v, v^+\rangle$, and one comes to the following formula for the visibility index, $\mathcal{F}(D) = \lim_{R\to\infty} \mathcal{F}_R(D)$:

$$\mathcal{F}(D) = \int_{(S^{d-1}\times S_r^{d-1})_-} f\Big(\arccos\langle v, v_D^+(v,\xi)\rangle\Big)\, d\mu(v,\xi), \tag{1}$$

with r being taken sufficiently large. The value of the integral in this formula does not depend on r (for the proof see Proposition 1.1 of Chapter 1 in the book [14]). The visibility index in this case is related to the situation when the distance to the background is much greater than the size of the domain itself.

Denote by $s_{d-1} = |S^{d-1}| = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ the area of the $(d-1)$-dimensional unit sphere, and by $b_d = \frac{2\pi^{d/2}}{d\Gamma(d/2)}$ the volume of the d-dimensional ball. One has, in particular, $s_0 = 2,\ s_1 = 2\pi,\ s_2 = 4\pi,\ b_1 = 2,\ b_2 = \pi,\ b_3 = 4\pi/3$.

Assume that

$$f(\phi) = c\,\phi^k(1 + o(1)) \quad \text{as } \phi \to 0^+ \quad (\text{with } c > 0,\ k > 0) \tag{2}$$

and introduce the notation

$$c_d = \begin{cases} c\,\frac{k^k}{(k+1)^{k+1}}\ \frac{\pi}{2^k}, & \text{if } d = 2 \\ c\,\frac{k^k}{(k+1)^{k+1}}\ \frac{s_{d-1}^{k+1} 2^{-1-dk+2k}}{\left(b_{d-1}s_{d-2}B\left(\frac{d-1}{2},\frac{d-2}{2}\right)\right)^k}, & \text{if } d \geq 3, \end{cases} \tag{3}$$

where B stands for the beta function, $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. In particular, in the 3D case we have $c_3 = c\,\frac{k^k}{(k+1)^{k+1}}\,\frac{1}{(2\pi)^{k-1}}$.

The following Theorem 1 establishes a connection between the visibility index of a domain, its volume, and the radius of a ball containing this domain.

Theorem 1. *Let a domain $D \subset \mathbb{R}^d$ be contained in a ball of radius r. Then its visibility index $\mathcal{F}(D)$, its volume $|D|$, and r are related by the inequality*

$$\frac{\mathcal{F}(D)}{r^{d-1}} \geq h_d(|D|/r^d),$$

where h_d is a function of a positive variable satisfying

$$h_d(x) = c_d\,x^{k+1}\,(1 + o(1)) \ \text{ as } x \to 0^+.$$

Note that the values $\mathcal{F}(D)/r^{d-1}$ and $|D|/r^d$ are preserved under a scaling transformation (applied to both D and the ambient ball). It is natural therefore that Theorem 1 relates these two values.

Remark 1. *It follows from Theorem 1 that*

$$\inf_{|D|=\text{const}} \mathcal{F}(D) \geq c_d\,|D|^{k+1}\frac{1}{r^{kd+1}}(1 + o(1)) \quad \text{as } r \to \infty.$$

This means that the infimum of visibility index in a class of domains with fixed volume contained in a certain ball is greater than a positive constant. This constant goes to zero when the radius of the ambient ball tends to infinity. It would be interesting to learn something about the upper bound for this infimum, at least for a special kind of the function f defining the visibility index. In particular, does the infimum go to zero as $r \to \infty$ (or, equivalently, as the diameter of D goes to infinity)? In other words, is it possible to construct a sequence of domains with fixed volume and with the visibility index going to zero?[1] *We do not know the answer to this question.*

One may wish to have a more direct estimate of the visibility index (without the term $o(1)$). We shall derive such estimates in the 2D and 3D cases for a particular choice of the function f. Namely, take $f(\theta) = 1 - \cos\theta$; the resulting visibility index

$$\mathfrak{F}(D) = \int_{(S^{d-1} \times S_r^{d-1})_-} \big(1 - \langle v, v_D^+(v, \xi) \rangle\big)\, d\mu(v, \xi)$$

(with r taken sufficiently large) has a simple mechanical interpretation in the framework of Newtonian aerodynamics [11]: it is just the mean value over all v of the aerodynamic resistance of D in the direction of v.[2] We shall call it the *mean resistance* of D.

Note that the mean resistance of a *convex* domain $C \subset \mathbb{R}^d$ can easily be determined; denoting by $|\partial C|$ the $(d-1)$-dimensional area of its boundary, one has $\mathfrak{F}(C) = \frac{4}{d+1}\, b_{d-1} |\partial C|$. In the 2D and 3D cases one has, respectively, $\mathfrak{F}(C) = 8|\partial C|/3$ and $\mathfrak{F}(C) = \pi |\partial C|$. In particular, the mean resistances of the 2D and 3D balls, B_r^2 and B_r^3, of radius r are equal, respectively, to $\mathfrak{F}(B_r^2) = \frac{16}{3}\pi r$ and $\mathfrak{F}(B_r^3) = 4\pi^2 r^2$. See sections 6.1.1 and 6.2 of [14] for details.

The following formulae for the mean resistance in the 2D and 3D cases are obtained from Theorem 1 by direct substitution $c = 1/2$, $k = 2$,

$$\frac{\mathfrak{F}(D)}{r} \ge h_2(|D|/r^2), \quad \text{where } h_2(x) = \frac{\pi}{54}\, x^3 (1 + o(1)), \ \ x \to 0^+ \quad \text{for } d = 2;$$

$$\frac{\mathfrak{F}(D)}{r^2} \ge h_3(|D|/r^3), \quad \text{where } h_3(x) = \frac{1}{27\pi}\, x^3 (1 + o(1)), \ \ x \to 0^+ \quad \text{for } d = 3.$$

The following theorem allows one to get rid of the term $o(1)$ in the above formulae.

[1] In this case the diameter of the domains should go to infinity.

[2] Of course it does not depend on the radius r of the ambient sphere and does not change when the domain is displaced within the sphere; see Proposition 1.1 of Chapter 1 in [14].

Theorem 2. *(a) Let a planar domain D with the area $|D|$ be contained in a circle of radius r. Then*

$$\frac{\mathfrak{F}(D)}{r} \ge \frac{\pi}{54}\left(\frac{|D|}{r^2}\right)^3.$$

(b) Let a 3-dimensional domain D with the volume $|D|$ be contained in a ball of radius r. Then

$$\frac{\mathfrak{F}(D)}{r^2} \ge \frac{1}{27\pi}\left(\frac{|D|}{r^3}\right)^3.$$

It is instructive to rewrite these formulas in terms of reduced volume κ_D and reduced resistance $\hat{\mathfrak{F}}_D$ defined by

$$\kappa_D = \frac{|D|}{b_d r^d}, \qquad \hat{\mathfrak{F}}_D = \frac{\mathfrak{F}(D)}{\frac{4}{d+1}\, b_{d-1} s_{d-1} r^{d-1}},$$

where r is the radius of the smallest ball containing D. One always has $0 < \kappa_D \le 1$, and $\kappa_D = 1$ iff D is a ball. In the latter case we have $\hat{\mathfrak{F}}_D = 1$. In the case $d = 2$ one has $\kappa_D = |D|/(\pi r^2)$ and $\hat{\mathfrak{F}}_D = \mathfrak{F}(D)/(\frac{16}{3}\,\pi r)$, and in the case $d = 3$ one has $\kappa_D = |D|/(\frac{4}{3}\,\pi r^3)$ and $\hat{\mathfrak{F}}_D = \mathfrak{F}(D)/(4\pi^2 r^2)$.

It is interesting to note that

$$\sup_{\kappa_D = \kappa} \hat{\mathfrak{F}}_D = \frac{d+1}{2}$$

for all $0 < \kappa < 1$. This can easily be derived from Theorem 6.2 in Chapter 6 of [14] by taking a sequence of domains inscribed in a certain ball and having the property of asymptotically perfect retro-reflection (that is, the initial velocity of the most part of incident particles is reversed as a result of reflections). Note that one can always ensure that the volume of each domain in the sequence equals κ by taking off the domain an appropriate smaller concentric ball.

On the contrary, only rough estimates are known for the infimum of $\hat{\mathfrak{F}}_D$. In particular, the following estimates in the 2D and 3D cases follow directly from Theorem 2,

$$\hat{\mathfrak{F}}_D \ge \frac{\pi^3}{288}\,\kappa_D^3 \ \text{ for } \ d = 2 \qquad \text{and} \qquad \hat{\mathfrak{F}}_D \ge \frac{16}{729}\,\kappa_D^3 \ \text{ for } \ d = 3. \tag{4}$$

These estimates are far from being sharp. Indeed, from the same Theorem 6.2 in [14] one can derive the exact value of the lower limit of $\hat{\mathfrak{F}}_D$ when $\kappa_D \to 1$; in particular,

$$\lim_{\tau \to 1^-} \inf_{\kappa_D = \tau} \hat{\mathfrak{F}}_D = m_2 \approx 0.987820 \ \text{ for } \ d = 2 \ \text{ and } \ \lim_{\tau \to 1^-} \inf_{\kappa_D = \tau} \hat{\mathfrak{F}}_D = m_3 \approx 0.969445 \ \text{ for } \ d = 3.$$

On the other hand, the values of the lower limits given by formulae (4) are much smaller, $\lim_{\tau\to 1^-} \inf_{\kappa_D=\tau} \hat{\mathfrak{F}}_D \ge \pi^3/288 \approx 0.11$ for $d = 2$ and $\lim_{\tau\to 1^-} \inf_{\kappa_D=\tau} \hat{\mathfrak{F}}_D \ge 16/729 \approx 0.022$ for $d = 3$.

There is a question on a natural generalization of formula (4). Consider the relative volume of a domain in its convex envelope, and let the normalized resistance be chosen so that the resistance of the convex envelope of the body equals 1. Is it possible to derive a sensible estimate for the normalized resistance in the spirit of formula (4)?

Remark 2. *The statement of Theorem 2 can also be interpreted in terms of Newton's problem of minimal resistance [11]. Consider a body moving in a rarefied medium of point particles. The medium is so rare that mutual interaction of particles is neglected, and particles are reflected elastically when hitting the body's boundary. One needs to find a body, from a prescribed class of bodies, that has the smallest aerodynamic resistance. There has been a significant progress in this problem in 1990s and 2000s (see, e.g., [2, 3, 9, 13, 14]).*

Suppose now that the body D translates in the medium and at the same time rotates (somersaults) very slowly and chaotically. In this case one is interested in minimizing the mean value of its resistance in all possible directions, i.e., the value $\mathfrak{F}(D)$. Theorem 2 gives a lower estimate of this value in a class of bodies with fixed volume.

Remark 3. *The statements of Theorems 1 and 2 hold for broader classes of dynamical systems than billiards. It suffices that the system satisfies the following conditions:*

(i) the motion is free outside a sphere of radius r;

(ii) all the trajectories of the system are continuous curves;

(iii) the standard measure $dv\,dx$ is invariant under the dynamics of the system.

In this case the proofs (given in the next section) go through without change.

One can, for example, take the free dynamics outside $D \subset \mathbb{R}^2$ with a pseudo-billiard law of reflection off the boundary ∂D; this law is induced by any one-to-one mapping of the segment $[-\pi/2, \pi/2]$ onto itself preserving the measure $\cos\varphi\, d\varphi$.

Remark 4. *Taking account of Jung's inequality between the diameter diam(D) of a set D and the smallest radius $r = r(D)$ of a ball containing D,*

$$r \le \sqrt{d/(2d+2)}\ \mathrm{diam}(D)$$

(see, e.g., subsection 2.6 in the book [5]), the inequalities in Theorem 2 can be replaced by the following (slightly weaker) ones,

$$\frac{\mathfrak{F}(D)}{\mathrm{diam}(D)} \ge \frac{\pi}{2\sqrt{3}} \left(\frac{|D|}{\mathrm{diam}(D)^2} \right)^3$$

in the 2D case, and

$$\frac{\mathfrak{F}(D)}{\operatorname{diam}(D)^2} \geq \frac{1}{\pi}\left(\frac{2^{21}}{3^{13}}\right)^{1/2}\left(\frac{|D|}{\operatorname{diam}(D)^3}\right)^3$$

in the 3D case. Substituting the exact values with approximate ones, one can then write down $\mathfrak{F}(D) > 0.9\,|D|^3\operatorname{diam}(D)^{-5}$ *in the 2D case, and* $\mathfrak{F}(D) > 0.36\,|D|^3\operatorname{diam}(D)^{-7}$ *in the 3D case.*

Further, using Jung's inequality and the obvious relation $\operatorname{diam}(D) \leq 2r$*, the inequality in Theorem 1 can be replaced with*

$$\frac{\mathcal{F}(D)}{\operatorname{diam}(D)^{d-1}} \geq 2^{1+\frac{d(k-1)}{2}}\left(\frac{d+1}{d}\right)^{\frac{d(k+1)}{2}} h_d\Big(\frac{|D|}{\operatorname{diam}(D)^d}\Big),$$

where h_d *is as in Theorem 1.*

3 Proofs of the theorems

All statements below are true up to subsets of measure zero.

Let us first prove Theorem 1. We consider the billiard inside the ball B_r^d of radius r and outside D. A particle starts moving at a point $\xi \in S_r^{d-1}$ and with a velocity $v \in S^{d-1}$ directed inside the sphere S^{d-1}, then makes several reflections off D, and finally intersects S^{d-1} again (at the point ξ^+ and with the velocity v^+) and disappears at the moment of intersection. The phase space of the billiard is $S^{d-1} \times (B_r^d \setminus D)$, and its volume V (with respect to the standard Liouville measure $dv\,dx$) equals

$$V = |S^{d-1} \times (B_r^d \setminus D)| = s_{d-1}(b_d r^d - |D|). \tag{5}$$

Denote by $l_{D,r}(v,\xi)$ the length of the billiard trajectory with the initial data $(v,\xi) \in (S^{d-1} \times S_r^{d-1})_-$ until the final intersection with S_r^{d-1}. We use Santaló-Stoyanov formula (see, e.g., [4, 22]), which in our case states that the phase volume is greater than or equal to the integral of the length of billiard trajectories over the initial data,

$$V \geq \int_{(S^{d-1}\times S_r^{d-1})_-} l_{D,r}(v,\xi)\,d\mu(v,\xi)$$

(note that the sign " $\geq$ " in this formula is due to the fact that a part of the phase space may be inaccessible for particles starting at the ambient sphere S_r^{d-1}). Writing ξ^+ in place of $\xi^+_{D,r}(v,\xi)$ for brevity and using the obvious inequality $l_{D,r}(v,\xi) \geq |\xi - \xi^+|$, we then obtain

$$V \geq \int_{(S^{d-1}\times S_r^{d-1})_-} |\xi - \xi^+|\,d\mu(v,\xi). \tag{6}$$

Further, take an orthonormal coordinate system, $x = (x_1, \dots x_d)$, centered at the origin and denote by $\mathcal{R}_v$ the rotation of $\mathbb{R}^d$ about the origin such that

(a) under this rotation, v goes to $(\bar{0},\, 1) := (0, \dots, 0, 1)$; that is, $\mathcal{R}_v v = (\bar{0}, 1)$;

(b) the 2-dimensional subspace of $\mathbb{R}^d$ spanned by the vectors v and $(\bar{0}, 1)$ is invariant under $\mathcal{R}_v$;

(c) $\mathcal{R}_v$ acts as identity on the orthogonal complement to this subspace.

Denote the upper and lower hemispheres of radius r by

$$\mathcal{S}_r^\pm = \{\eta = (\eta_1, \dots, \eta_d) \in S_r^{d-1} : \pm\eta_d \geq 0\}.$$

Denote $\eta' = (\eta_1, \dots, \eta_{d-1})$ and consider the standard measure $d\eta'$ on both the hemispheres (that is, the measure of a Borel subset in $\mathcal{S}_r^\pm$ is equal to the Lebesgue measure of its orthogonal projection on the subspace $\eta_d = 0$). For each choice of the sign " $+$ " or " $-$ " and for all $v \in S^{d-1}$, the mapping $\xi \mapsto \mathcal{R}_v \xi$ from the hemisphere $\{\xi \in S_r^{d-1} : \pm\langle v, \xi\rangle \geq 0\}$ with the measure $|\langle v, n(\xi)\rangle|\, d\xi$ onto the hemisphere $\mathcal{S}_r^\pm$ with the measure $d\eta'$ is measure preserving.

It follows (by Cavalieri's principle) that the bijective mapping $(v, \xi) \mapsto (v, \mathcal{R}_v \xi)$ between the space $(S^{d-1} \times S_r^{d-1})_\pm$ with the measure μ (recall that it is defined by $d\mu(v, \xi) = |\langle v,\, n(\xi)\rangle|\, dv\, d\xi$) and the space $S^{d-1} \times \mathcal{S}_r^\pm$ with the measure defined by $dv\, d\eta'$ is also measure preserving. This implies, in particular, that

$$\mu((S^{d-1} \times S_r^{d-1})_-) = \mu((S^{d-1} \times S_r^{d-1})_+) = |S^{d-1}|\, |B_r^{d-1}| = s_{d-1} b_{d-1} r^{d-1}.$$

Now define the measures $\mu_\pm$ on $\mathcal{S}_r^\pm$ by $d\mu_\pm = s_{d-1} d\eta'$; then the mappings $\pi_\pm : (S^{d-1} \times S_r^{d-1})_\pm \to \mathcal{S}_r^\pm$ defined by

$$\pi_\pm(v, \xi) = \mathcal{R}_v \xi$$

are measure preserving.

We have

$$|\xi - \xi^+| = |\mathcal{R}_v \xi - \mathcal{R}_v \xi^+| \geq |\mathcal{R}_v \xi - \mathcal{R}_{v^+} \xi^+| - |\mathcal{R}_v \xi^+ - \mathcal{R}_{v^+} \xi^+|. \tag{7}$$

Fix a value $0 < \phi < \pi$ and let Σ_ϕ be the set of values $(v, \xi) \in (S^{d-1} \times S_r^{d-1})_-$ such that the angle between v and $v^+ = v_D^+(v, \xi)$ is greater or equal than ϕ; that is,

$$\Sigma_\phi = \{(v, \xi) \in (S^{d-1} \times S_r^{d-1})_- : \langle v, v^+\rangle \leq \cos\phi\}. \tag{8}$$

From (6) and (7), taking into account that $|\mathcal{R}_v \xi^+ - \mathcal{R}_{v^+} \xi^+| \leq 2r$, we get

$$V \geq \int_{(S^{d-1} \times S_r^{d-1})_-} |\mathcal{R}_v \xi - \mathcal{R}_{v^+} \xi^+|\, d\mu(v, \xi) - \int_{(S^{d-1} \times S_r^{d-1})_-} |\mathcal{R}_v \xi^+ - \mathcal{R}_{v^+} \xi^+|\, d\mu(v, \xi)$$

$$\geq \int_{(S^{d-1}\times S_r^{d-1})_-} |\mathcal{R}_v\xi - \mathcal{R}_{v^+}\xi^+|\, d\mu(v,\xi)$$

$$- \int_{(S^{d-1}\times S_r^{d-1})_-\setminus\Sigma_\phi} |\mathcal{R}_v\xi^+ - \mathcal{R}_{v^+}\xi^+|\, d\mu(v,\xi) - 2r\,\mu(\Sigma_\phi). \quad (9)$$

Let us estimate the three terms in the right hand side of (9).

3.1. Denote for brevity $X = (S^{d-1}\times S_r^{d-1})_-$ and recall that $f(\phi)$ is positive and monotone increasing for $0<\phi<\pi$. By Chebyshev's inequality for $t>0$ one has

$$\mu\Big(\big\{(v,\xi)\in X : f(\arccos\langle v, v_D^+(v,\xi)\rangle) \geq t\big\}\Big) \leq \frac{1}{t}\int_X f\big(\arccos\langle v, v_D^+(v,\xi)\rangle\big)\, d\mu(v,\xi).$$

Substituting $t = f(\phi)$, using the definitions (1) and (8), and multiplying both parts of the inequality by $2r$ one obtains

$$2r\,\mu(\Sigma_\phi) \leq 2r\,\frac{\mathcal{F}(D)}{f(\phi)}. \quad (10)$$

3.2. If $(v,\xi)\in(S^{d-1}\times S_r^{d-1})_-\setminus\Sigma_\phi$ then the angle between v and v^+ is less than ϕ, and denoting by $\alpha=\alpha(v)$ and $\alpha^+=\alpha(v^+)$ the angles formed by the vectors v and v^+ with $(\bar{0},1)$, $0\leq\alpha,\ \alpha^+\leq\pi$; we have $|\alpha-\alpha^+|\leq\phi$.

In the case $d=2$ one obviously has

$$|\mathcal{R}_v\xi^+ - \mathcal{R}_{v^+}\xi^+| \leq 2r\sin\frac{\phi}{2}.$$

In the case $d\geq 3$ the estimate is more difficult.

Consider the 3-dimensional subspace of $\mathbb{R}^d$ spanned by the vectors $v,\ v^+$ and $(\bar{0},1)$. The restrictions of $\mathcal{R}_v$ and $\mathcal{R}_{v^+}$ on this subspace are rotations by the angles α and α^+, respectively. Let w and w^+ be unit vectors in this subspace pointing at directions of the rotation axes. Both w and w^+ are orthogonal to $(\bar{0},1)$. The restriction of $\mathcal{R}_{v^+}^{-1}\mathcal{R}_v$ on this subspace acts as a rotation by an angle β, and its restriction on the orthogonal complement to this subspace is an identity. We have

$$|\mathcal{R}_v\xi^+ - \mathcal{R}_{v^+}\xi^+| = |\mathcal{R}_{v^+}^{-1}\mathcal{R}_v\xi^+ - \xi^+| \leq 2r\sin\frac{\beta}{2},$$

therefore we need to estimate $\sin\frac{\beta}{2}$. To that end we shall proceed to some trigonometric calculations.

Introduce an orthonormal coordinate system x, y, z in the chosen subspace, where the third coordinate axis coincides with the dth axis of the original space $\mathbb{R}^d$ and the origin coincides with the origin in the space $\mathbb{R}^d$. In this system the coordinate vectors v, v^+, and $(\bar{0}, 1)$ take the form

$$(\sin\alpha\cos\theta,\ \sin\alpha\sin\theta,\ \cos\alpha); \qquad (\sin\alpha^+\cos\theta^+,\ \sin\alpha^+\sin\theta^+,\ \cos\alpha^+); \quad (0,\ 0,\ 1).$$

One has

$$\langle v, v^+\rangle = \cos\alpha\,\cos\alpha^+ + \sin\alpha\,\sin\alpha^+\cos(\theta-\theta^+). \tag{11}$$

Further, one easily finds that $w = (-\sin\theta,\ \cos\theta,\ 0)$, $w^+ = (-\sin\theta^+,\ \cos\theta^+,\ 0)$, and therefore

$$\langle w, w^+\rangle = \cos(\theta-\theta^+). \tag{12}$$

Taking into account that $\langle v, v^+\rangle \geq \cos\phi$ and using (11) and (12), one finds

$$\langle w, w^+\rangle \geq \frac{\cos\phi - \cos\alpha\,\cos\alpha^+}{\sin\alpha\,\sin\alpha^+}. \tag{13}$$

In what follows we shall use the same notation $\mathcal{R}_v$ and $\mathcal{R}_{v^+}$ for the restrictions of the corresponding rotations on our 3D subspace. It is convenient to represent them in the quaternionic form: $\mathcal{R}_v$ is the action $u \mapsto quq^{-1}$ of the quaternion

$$q = \cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\ w,$$

and $\mathcal{R}_{v^+}$ is the action $u \mapsto q_+uq_+^{-1}$ of the quaternion

$$q_+ = \cos\frac{\alpha^+}{2} + \sin\frac{\alpha^+}{2}\ w^+.$$

Correspondingly, $\mathcal{R}_{v^+}^{-1}\mathcal{R}_v$ is the action of the quaternion

$$q_+^{-1}q = \Big(\cos\frac{\alpha^+}{2} - \sin\frac{\alpha^+}{2}\ w^+\Big)\Big(\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\ w\Big)$$

$$= \Big[\cos\frac{\alpha}{2}\cos\frac{\alpha^+}{2} + \sin\frac{\alpha}{2}\sin\frac{\alpha^+}{2}\langle w, w^+\rangle\Big] + \Big[\cos\frac{\alpha^+}{2}\sin\frac{\alpha}{2}\ w - \cos\frac{\alpha}{2}\sin\frac{\alpha^+}{2}\ w^+ + w^+\times w\rangle\Big],$$

which has the real part

$$\cos\frac{\beta}{2} = \cos\frac{\alpha}{2}\cos\frac{\alpha^+}{2} + \sin\frac{\alpha}{2}\sin\frac{\alpha^+}{2}\langle w, w^+\rangle.$$

From this formula, using (13) and taking into account the double angle formulas for sine and cosine, one obtains the estimate

$$\cos\frac{\beta}{2} \ge \cos\frac{\alpha}{2}\ \cos\frac{\alpha^+}{2} + \frac{\cos\phi - \cos\alpha\,(2\cos^2\frac{\alpha^+}{2} - 1)}{4\cos\frac{\alpha}{2}\ \cos\frac{\alpha^+}{2}}.$$

Denoting $\cos\frac{\alpha^+}{2} =: z$, one comes to the inequality

$$\cos\frac{\beta}{2} \ge \inf_{0\le z\le 1}\left(z\cos\frac{\alpha}{2} + \frac{\cos\phi - \cos\alpha\,(2z^2-1)}{4z\cos\frac{\alpha}{2}}\right).$$

If $0 < \alpha < \pi - \phi$, the infimum of the expression in the brackets is attained at $z = \sqrt{\cos\phi + \cos\alpha}/\sqrt{2}$. Substituting this value in the latter inequality, one gets

$$\cos\frac{\beta}{2} \ge \frac{\sqrt{\cos\phi + \cos\alpha}}{\sqrt{2}\,\cos\frac{\alpha}{2}}.$$

From here after some algebra one finally obtains

$$\sin\frac{\beta}{2} \le \frac{\sin\frac{\phi}{2}}{\cos\frac{\alpha}{2}}.$$

Note that the right hand side in this inequality is smaller than 1.

Thus, we have

$$|\mathcal{R}_v\xi^+ - \mathcal{R}_{v^+}\xi^+| = |\mathcal{R}_{v^+}^{-1}\mathcal{R}_v\xi^+ - \xi^+| \le 2r\,h_\phi(\alpha),$$

where

$$h_\phi(\alpha) = \begin{cases} \frac{\sin\frac{\phi}{2}}{\cos\frac{\alpha}{2}}, & \text{if } 0 \le \alpha < \pi - \phi \\ 1, & \text{if } \pi - \phi \le \alpha \le \pi \end{cases}$$

in the case $d \ge 3$, and $h_\phi(\alpha) = \sin\frac{\phi}{2}$ in the case $d = 2$. We now have an estimate for the second integral in the right hand side of (9)

$$\int_{(S^{d-1}\times S_r^{d-1})_-\setminus\Sigma_\phi} |\mathcal{R}_v\xi^+ - \mathcal{R}_{v^+}\xi^+|\,d\mu(v,\xi) \le 2r\int_{(S^{d-1}\times S_r^{d-1})_-} h_\phi(\alpha(v))\,d\mu(v,\xi).$$

We now need to estimate the integral in the right hand side of this inequality. Integrating by ξ gives us the factor $b_{d-1}r^{d-1}$. Integrating by v over S^{d-1} amounts to integration with the differential $s_{d-2}\sin^{d-2}\alpha\,d\alpha$ over the interval $\alpha \in [0,\,\pi]$. Thus we get

$$2r\int_{(S^{d-1}\times S_r^{d-1})_-} h_\phi(\alpha(v))\,d\mu(v,\xi) = 2r^2 b_1 s_0\int_0^\pi \sin\frac{\phi}{2}\,d\alpha = 8r^2\pi\,\sin\frac{\phi}{2}$$

in the case $d=2$ and

$$2r\int_{(S^{d-1}\times S_r^{d-1})_-} h_\phi(\alpha(v))\,d\mu(v,\xi) = 2r^d\,b_{d-1}s_{d-2}\Big(\sin\frac{\phi}{2}\int_0^{\pi-\phi}\frac{\sin^{d-2}\alpha}{\cos\frac{\alpha}{2}}\,d\alpha+\int_{\pi-\phi}^{\pi}\sin^{d-2}\alpha\,d\alpha\Big)$$

in the case $d\ge 3$.

Introducing the functions $\mathcal{I}_d(\phi)$, $\phi\in[0,\,\pi]$, $d=2,3,\dots$ by

$$\mathcal{I}_2(\phi)=\pi\sin\frac{\phi}{2}$$

and

$$\mathcal{I}_d(\phi)=\sin\frac{\phi}{2}\int_0^{\pi-\phi}\frac{\sin^{d-2}\alpha}{\cos\frac{\alpha}{2}}\,d\alpha+\int_{\pi-\phi}^{\pi}\sin^{d-2}\alpha\,d\alpha \tag{14}$$

for $d\ge 3$, we can now write

$$\int_{(S^{d-1}\times S_r^{d-1})_-\setminus\Sigma_\phi}|\mathcal{R}_v\xi^+-\mathcal{R}_{v^+}\xi^+|\,d\mu(v,\xi)\le 2r^d\,b_{d-1}s_{d-2}\,\mathcal{I}_d(\phi). \tag{15}$$

For small values of ϕ we have the following asymptotic behavior:

$$\mathcal{I}_2(\phi)=\frac{\pi}{2}\phi(1+o(1))\ \text{ and }\ \mathcal{I}_d(\phi)=2^{d-3}B\Big(\frac{d-1}{2},\frac{d-2}{2}\Big)\,\phi(1+o(1))\ (d\ge 3)\text{ as }\phi\to 0^+. \tag{16}$$

The function $\mathcal{I}_3$ can easily be calculated in the case $d=3$,

$$\mathcal{I}_3(\phi)=4\sin\frac{\phi}{2}-2\sin^2\frac{\phi}{2}.$$

3.3. Now consider the first term in the right hand side of (9).

Recall that the mapping $T=T_{D,r}:(S^{d-1}\times S_r^{d-1})_-\to(S^{d-1}\times S_r^{d-1})_+$ is defined by $T(v,\xi)=\big(v_D^+(v,\xi),\xi_{D,R}^+(v,\xi)\big)$. It preserves the measure μ, and therefore induces a measure on $(S^{d-1}\times S_r^{d-1})_-\times(S^{d-1}\times S_r^{d-1})_+$ concentrated on the graph of T and whose projections on $(S^{d-1}\times S_r^{d-1})_-$ and $(S^{d-1}\times S_r^{d-1})_+$ coincide with μ. The push forward of this measure under the map[3] $\pi_-\times\pi_+:(S^{d-1}\times S_r^{d-1})_-\times(S^{d-1}\times S_r^{d-1})_+\to\mathcal{S}_r^-\times\mathcal{S}_r^+$ (let it be denoted by $\nu_{D,r}$) is a measure on $\mathcal{S}_r^-\times\mathcal{S}_r^+$ whose projections on $\mathcal{S}_r^-$ and on $\mathcal{S}_r^+$ coincide, respectively, with μ_- and μ_+.

[3]this map sends (v,ξ,v^+,ξ^+) to $(\mathcal{R}_v\xi,\,\mathcal{R}_{v^+}\xi^+)$

Therefore we have

$$\int_{(S^{d-1}\times S_r^{d-1})_-} |\mathcal{R}_v\xi - \mathcal{R}_{v^+}\xi^+|\, d\mu(v,\xi) = \int_{\mathcal{S}_r^-\times\mathcal{S}_r^+} |\eta-\eta^+|\, d\nu_{D,r}(\eta,\eta^+)$$

$$\geq \inf_\nu \int_{\mathcal{S}_r^-\times\mathcal{S}_r^+} |\eta-\eta^+|\, d\nu(\eta,\eta^+),$$

where the infimum is taken over all measures ν whose projections on $\mathcal{S}_r^-$ and on $\mathcal{S}_r^+$ coincide with μ_- and μ_+. This problem of searching for the minimum is actually a problem of optimal mass transport, which in this case is easy to solve. We use the inequality $|\eta-\eta^+| \geq |\eta_d-\eta_d^+| = \eta_d^+ - \eta_d$ (since $\eta_d = -\sqrt{r^2-\sum_{i=1}^{d-1}\eta_i^2} \leq 0$ and $\eta_d^+ = \sqrt{r^2-\sum_{i=1}^{d-1}(\eta_i^+)^2} \geq 0$) to get

$$\int_{\mathcal{S}_r^-\times\mathcal{S}_r^+} |\eta-\eta^+|\, d\nu_D(\eta,\eta^+) \geq \int_{\mathcal{S}_r^-\times\mathcal{S}_r^+} \eta_d^+\, d\nu_D(\eta,\eta^+) - \int_{\mathcal{S}_r^-\times\mathcal{S}_r^+} \eta\, d\nu_D(\eta,\eta^+)$$

$$= \int_{\mathcal{S}_r^+} \eta_d^+\, d\mu_+(\eta^+) - \int_{\mathcal{S}_r^-} \eta_d\, d\mu_-(\eta) = \int_{B_r^{d-1}} \Big(r^2 - \sum_{i=1}^{d-1}(\eta_i^+)^2\Big)^{1/2} s_{d-1}\; d\eta_1^+\ldots d\eta_{d-1}^+$$

$$-\int_{B_r^{d-1}} \Big[-\Big(r^2-\sum_{i=1}^{d-1}\eta_i^2\Big)^{1/2}\Big] s_{d-1}\, d\eta_1\ldots d\eta_{d-1} = s_{d-1}b_d\, r^d.$$ [4]

That is, we have

$$\int_{(S^{d-1}\times S_r^{d-1})_-} |\mathcal{R}_v\xi - \mathcal{R}_{v^+}\xi^+|\, d\mu(v,\xi) \geq s_{d-1}b_d\, r^d. \tag{17}$$

3.4. From (5), (9), (10), (15), and (17) we obtain

$$s_{d-1}b_d\, r^d - s_{d-1}|D| \geq s_{d-1}b_d\, r^d - 2r^d\, b_{d-1}s_{d-2}\,\mathcal{I}_d(\phi) - 2r\,\frac{\mathcal{F}(D)}{f(\phi)}.$$

It follows that

$$|D| \leq \inf_{0<\phi<\pi}\Big(\frac{2r\,\mathcal{F}(D)}{s_{d-1}}\,\frac{1}{f(\phi)} + \frac{2b_{d-1}s_{d-2}r^d}{s_{d-1}}\,\mathcal{I}_d(\phi)\Big). \tag{18}$$

[4] This value is really attained at the (optimal) measure supported on the subspace $(\eta_1,\ldots,\eta_{d-1}) = (\eta_1^+,\ldots,\eta_{d-1}^+)$. This measure induces the mass transport in the vertical direction sending each point of the lower hemisphere $\mathcal{S}_r^-$ to the point of the upper hemisphere $\mathcal{S}_r^+$ with the same abscissa.

Using asymptotic formulae (2) and (16) for f and $\mathcal{I}_d$, respectively, and replacing both terms in the right hand side of (18) with their approximated values (as $\phi \to 0^+$), we obtain the expression

$$\frac{\alpha}{k}\,\phi^{-k} + \beta\phi, \tag{19}$$

where

$$\alpha = \frac{2kr\,\mathcal{F}(D)}{cs_{d-1}} \quad \text{and} \quad \beta = \begin{cases} 2r^2, & \text{if } d = 2 \\ \frac{1}{s_{d-1}} 2^{d-2} b_{d-1} s_{d-2} B\big(\frac{d-1}{2}, \frac{d-2}{2}\big) r^d, & \text{if } d \geq 3. \end{cases}$$

The minimum of (19) is equal to $\frac{k+1}{k}\,\alpha^{1/(k+1)}\beta^{k/(k+1)}$ and is attained at $\phi_* = (\alpha/\beta)^{1/(k+1)}$. Substituting this value ϕ_* in the right hand side of (18) and raising both parts of the resulting inequality to the $(k+1)$th power, one obtains

$$\left(\frac{|D|}{r^d}\right)^{k+1} \leq \frac{1}{c_d}\,\frac{\mathcal{F}(D)}{r^{d-1}}\,(1 + o(1)), \tag{20}$$

where c_d is defined by (3) and $o(1)$ means a function of $\mathcal{F}(D)/r^{d-1}$ vanishing when its argument goes to zero. Reversing relation (20), one gets

$$\frac{\mathcal{F}(D)}{r^{d-1}} \geq c_d \left(\frac{|D|}{r^d}\right)^{k+1} (1 + o(1));$$

this time $o(1)$ means a function of $|D|/r^d$ vanishing when its argument goes to zero. Theorem 1 is proved.

Let us now prove Theorem 2. Here we have $f(\phi) = 1 - \cos\phi$ and use the notation $\mathfrak{F}$ in place of $\mathcal{F}$ in this particular case.

If $d = 2$, substitute $s_1 = 2\pi$, $b_1 = 2$, $s_0 = 2$, and $\mathcal{I}_2(\phi) = \pi\sin\frac{\phi}{2}$ into (18) to obtain

$$|D| \leq \inf_{0<\phi<\pi}\left(\frac{r\mathfrak{F}(D)}{2\pi\sin^2\frac{\phi}{2}} + 4r^2\sin\frac{\phi}{2}\right).$$

Introducing the shorthand notation $z = \sin\frac{\phi}{2}$, $A = |D|$, $\mathfrak{F} = \mathfrak{F}(D)$, this inequality can be rewritten as

$$A \leq \inf_{0<z<1}\left(\frac{r\mathfrak{F}}{2\pi z^2} + 4r^2 z\right). \tag{21}$$

Our goal is to prove the inequality

$$A^3 \leq \frac{54}{\pi}\,r^5\mathfrak{F}, \tag{22}$$

which is equivalent to statement (a) of Theorem 2.

Consider two cases. If $\mathfrak{F} \le 4\pi r$, the infimum in (21) is attained at $z_* = (\mathfrak{F}/(4\pi r))^{1/3}$, and substituting z_* in (21), we get (22). On the other hand, if $\mathfrak{F} > 4\pi r$, we obviously have (since D is contained in a circle of radius r)

$$A^3 \le (\pi r^2)^3 < (6r^2)^3 = \frac{54}{\pi}\, r^5 \cdot 4\pi r < \frac{54}{\pi}\, r^5 \mathfrak{F},$$

and we again come to (22). Thus, statement (a) of Theorem 2 is proved.

If $d = 3$, one has $s_2 = 4\pi$, $b_2 = \pi$, $s_1 = 2\pi$, and $\mathcal{I}_3(\phi) = 4\sin\frac{\phi}{2} - 2\sin^2\frac{\phi}{2}$, and inequality (18) takes the form

$$|D| \le \inf_{0<\phi<\pi} \Big[\frac{r\mathfrak{F}(D)}{4\pi \sin^2\frac{\phi}{2}} + 2\pi r^3\Big(2\sin\frac{\phi}{2} - \sin^2\frac{\phi}{2}\Big)\Big].$$

Introducing the notation $\tilde{A} = |D|/(2\pi r^3)$, $\tilde{\mathfrak{F}} = \mathfrak{F}(D)/(8\pi^2 r^2)$, and $z = \sin\frac{\phi}{2}$, one rewrites the last inequality in the form

$$\tilde{A} \le \inf_{0\le z\le 1} h(z), \quad \text{where } h(z) = \frac{\tilde{\mathfrak{F}}}{z^2} + 2z - z^2. \tag{23}$$

We are going to prove the inequality

$$\tilde{A}^3 \le 27\tilde{\mathfrak{F}}, \tag{24}$$

which is equivalent to statement (b) of Theorem 2.

After a simple algebra one concludes that if $\tilde{\mathfrak{F}} > 27/256$, we have $h'(z) < 0$ for all $z > 0$. has a unique zero $z = 3/4$. If $0 < \tilde{\mathfrak{F}} \le 27/256$, the equation $h'(z) = 0$ has two positive zeros (coinciding when $\tilde{\mathfrak{F}} = 27/256$). The smallest zero $z_* = z_*(\tilde{\mathfrak{F}})$ (which is a local minimizer of h if $\tilde{\mathfrak{F}}$ is strictly smaller than 27/256) satisfies the inequality $0 < z_* \le 3/4$. It is also straightforward to check that

$$\tilde{\mathfrak{F}} = z_*^3(1 - z_*). \tag{25}$$

Consider two cases. If $\tilde{\mathfrak{F}} \le 27/256$, we substitute $z = z_*$ in (23) and use (25) to obtain $\tilde{A} \le 3z_*(1-2z_*/3)$. Taking the third power of both sides of this inequality and using that $(1-2z/3)^3 < 1 - z$ for $0 < z \le 3/4$, we come to (24). If, otherwise, $\tilde{\mathfrak{F}} > 27/256$, we use that $|D| \le \frac{4}{3}\pi r^3$, and therefore $\tilde{A} \le 2/3$. It follows that $\tilde{A}^3 \le 8/27 < 27 \cdot 27/256 < 27\tilde{\mathfrak{F}}$, and (24) again follows. Thus, statement (b) of Theorem 2 is also proved.

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